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A THEOREM IN FACTORIALS.

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Let the indicated product

$$a(a-d)(a-2d) \dots [a-(n-1)d]$$

be expanded by multiplication into the series

$$a^n + \varphi_1(n)a^{n-1}d + \varphi_2(n)a^{n-2}d^2 + \dots + \varphi_{n-2}(n)a^2d^{n-2} + \varphi_{n-1}(n)ad^{n-1},$$

$\varphi_1(n)$, $\varphi_2(n)$, \dots being functions of n whose form does not depend on the value of n .

Let the symbol ${}_da_n$, whatever the values of a , d , and n , be defined by

$${}_da_n \equiv a^n + \varphi_1(n)a^{n-1}d + \varphi_2(n)a^{n-2}d^2 + \dots \quad (A)$$

Then let us consider the following

$$\begin{aligned} \text{THEOREM.—} {}_d(x+y)_n &= {}_dx_n + n \cdot {}_dx_{n-1} \cdot {}_dy_1 + \frac{n(n-1)}{2!} \cdot {}_dx_{n-2} \cdot {}_dy_2 \\ &\quad + \frac{n(n-1)(n-2)}{3!} \cdot {}_dx_{n-3} \cdot {}_dy_3 + \dots \end{aligned}$$

I. Let n be a positive integer.

In this case, from (A),

$${}_d(x+y)_n = (x+y)(x+y-d)(x+y-2d) \dots [x+y-(n-1)d],$$

$${}_dx_n = x(x-d)(x-2d) \dots [x-(n-1)d],$$

$${}_dx_{n-1} = x(x-d)(x-2d) \dots [x-(n-2)d],$$

$$\dots \dots \dots$$

Let us assume the theorem true for any particular value of n , and multiply both sides by $x+y-nd$. The left side obviously becomes ${}_d(x+y)_{n+1}$. In multiplying the $(r+1)$ th term of the right side, separate $x+y-nd$ into $x-(n-r)d$ and $y-rd$. The $(r+1)$ th term,

$$\frac{n(n-1)(n-2) \dots (n-r+1)}{r!} \cdot {}_dx_{n-r} \cdot {}_dy_r,$$

multiplied by $x-(n-r)d$ equals

$$\frac{n(n-1)(n-2) \dots (n-r+1)}{r!} \cdot {}_dx_{n-r+1} \cdot {}_dy_r,$$

and multiplied by $y - rd$ equals

$$\frac{n(n-1)(n-2) \dots (n-r+1)}{r!} \cdot {}_d x_{n-r} \cdot {}_d y_{r+1}.$$

The r th term treated in like manner will yield the two products

$$\frac{n(n-1)(n-2) \dots (n-r+2)}{(r-1)!} \cdot {}_d x_{n-r+2} \cdot {}_d y_{r-1}$$

and

$$\frac{n(n-1)(n-2) \dots (n-r+2)}{(r-1)!} \cdot {}_d x_{n-r+1} \cdot {}_d y_r.$$

The first of the first pair of products and the second of the second pair are the only terms of the product of the right side by $x + y - nd$ in which ${}_d x_{n-r+1} \cdot {}_d y_r$ will appear. The sum of these will therefore be one term of the complete product; viz. the $(r+1)$ th term. But this is identical with the $(r+1)$ th term in the expansion of ${}_d(x+y)_{n+1}$ by the theorem. Hence if the theorem is true for any particular value of n , it is true for the value greater by unity; but it is evidently true when $n=1$; it is therefore true for any positive integral value of n .

As a particular case, when $d=1$, we have Vandermonde's Theorem.

Again, when $d=0$, we have the Binomial Theorem for the case of a positive integral exponent, since, from (A), ${}_0 a_n = a^n$.

II.—Let n be a negative integer.

Let $n = -m$. We will first show that

$${}_d a_{-m} = \frac{1}{{}_d(a+md)_m}.$$

Now

$${}_d a_r = {}_d a_{r-1} \cdot [a - (r-1)d]; \quad (1)$$

for ${}_d a_{r-1}$ expanded by (A) and multiplied by $a - (r-1)d$ becomes the expansion of ${}_d a_r$ by (A). Hence,

$${}_d a_r = {}_d a_{r-s} \cdot {}_d [a - (r-s)d]_s, \quad (2)$$

s being a positive integer. Now from (A) ${}_d a_1 = a$. From (1) ${}_d a_1 = {}_d a_0 \cdot a$. Hence ${}_d a_0 = 1$. From (2) ${}_d a_0 = {}_d a_{-s} \cdot (a + sd)_s$. Hence

$${}_d a_{-m} = {}_d (a + md)_m^{-1} \quad (3)$$

Let $E_d(x+y)_n$ denote the expansion of ${}_d(x+y)_n$ by the theorem, and let $E_d(x+y)_{-m}$ be multiplied by $E_d(x+y+md)_m$. If $E_d(x+y)_{-m}$ is convergent and remains convergent when its negative terms (if there are any) are made positive, the series resulting from multiplication will be convergent, and will equal ${}_d(x+y+md)_m$ multiplied by the value of $E_d(x+y)_{-m}$.*

* See Cauchy's *Analyse Algébrique*, or Chas. Smith's *Treatise on Algebra*.

But $E_d(x+y)_{-m} \times E_d(x+y+md)_m = 1$,

and $_d(x+y)_{-m} \times _d(x+y+md)_m = 1$;

hence $_d(x+y)_{-m} = E_d(x+y)_{-m}$

under the condition above stated.

It may be shown that $E_d(x+y)_{-m}$ is convergent when y is numerically less than $x+d$.

It follows from the above that $E_d(x+y)_{-m}$ is never finite when $_d(x+y)_{-m}$ is infinite.

We can thence prove the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$$

divergent; for

$$E_1(0-1)_{-1} = {}_1O_{-1} + (-1) \cdot {}_1O_{-2} \cdot {}_1(-1)_1 + \frac{(-1)(-2)}{2!} \cdot {}_1O_{-3} \cdot {}_1(-1)_2 + \dots$$

$$= \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1 \cdot 2}{1 \cdot 2 \cdot 3} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \dots,$$

and

$${}_1(0-1)_{-1} = {}_1O_1^{-1} = O^{-1} = \infty.$$

Again, the familiar series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$$

is

$$E_1(1-1)_{-1}, \text{ and } {}_1(1-1)_{-1} = {}_1I_1^{-1} = 1.$$

The theorem is thus seen to be of use in the summation of series.

When $d=0$ we have here the Binomial Theorem for the case of a negative integral exponent.

III. *Let n be any number.*

1. Let $d=0$. The theorem becomes the Binomial Theorem for any index, which has been proved true when the series is convergent; and the series has been proved convergent when y is numerically less than x . We will therefore assume these results.

2. Let d be any number. Let $D_d(x+y)_n$ denote the expansion of $_d(x+y)_n$ by (A). If y be numerically less than x , we can substitute for $(x+y)^n$, $(x+y)^{n-1}$, \dots in $D_d(x+y)_n$ their respective expansions by the Binomial Theorem. Then the coefficient of any power of x in $D_d(x+y)_n$ equals the coefficient of the same power of x in $E_d(x+y)_n$. Hence the theorem is true for any value of n , provided y is numerically less than x .